$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^x - 2y}{2\cos 2z - x}.$$

19) $L.H.S. = \frac{1}{x} z_x + \frac{1}{y} z_y = \frac{1}{x} * y\varphi'(x^2-y^2) * 2x$

$$+\frac{1}{y}\left[\varphi(x^2-y^2) + y\varphi'(x^2-y^2) * (-2y)\right]$$

$$= 2y\varphi' + \frac{1}{y}\varphi - 2y\varphi' = \frac{1}{y}\varphi(x^2-y^2)$$

$$= \frac{y\varphi(x^2-y^2)}{y^2} = \frac{z}{y^2} = R.H.S.$$

-o-o-o-

ثانيا : مسائل يحلها الطالب بنفسه

أوجد المشتقات الجزئية الموضحة أمام كل دالة :

1) $z = xt^3 + y^2 \ln(t + 2x)$ ؛ $\frac{\partial z}{\partial x}$ ، $\frac{\partial z}{\partial y}$ ، $\frac{\partial z}{\partial t}$

$$\frac{\partial Z}{\partial X} = t^3 + \frac{2Y^2}{t+2X}$$

$$\frac{\partial Z}{\partial Y} = 2Y \ln(t+2X)$$

$$\frac{\partial Z}{\partial t} = 3Xt^2 + \frac{Y^2}{t+2X}$$

2) $u = \tan^{-1}\frac{y}{x}$; $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial u}{\partial x} = \frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial^2 y^2} = \frac{0-2xy}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

3) $z = x^y$; $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$

$$\frac{\partial z}{\partial y} = x^y \ln x \Longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{x^y}{x} + y\left(x^{y-1}\right)\ln x$$

$$\frac{\partial z}{\partial x} = y x^{y-1} \Longrightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

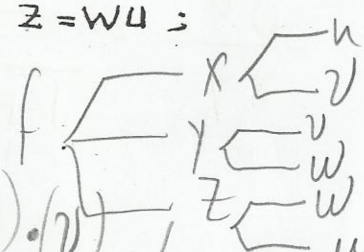
$$\therefore \frac{\partial^2 z}{\partial y \partial x} = x^{y-1} + y\left(x^{y-1}\right)\ln x$$

$$\frac{\partial u}{\partial x} = \frac{\frac{1+x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}} = \frac{1+x}{x\sqrt{x^2+y^2}+x^2+y^2}$$

4) $u = \ln(x+\sqrt{x^2+y^2})$; u_{xx}

$$u_{xx} = \frac{x\sqrt{x^2+y^2}+x^2+y^2-(1+x)\left[\frac{4x^3+2y^2x}{2\sqrt{x^4+y^2x^2}}+2x\right]}{\left(x\sqrt{x^4+y^2x^2}+x^2+y^2\right)^2}$$

5) $f = z \sin xy$; $x = uv$, $y = vw$, $z = wu$;
f_u , f_v , f_w

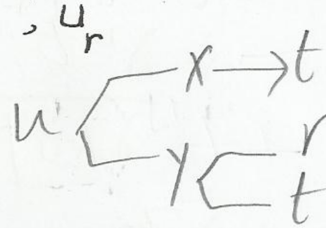


$$f_u = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u} = (yz\cos xy)\cdot(v) + (\sin xy)(w)$$

$$f_v = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = (yz\cos xy)(u) + (xz\cos xy)(w)$$

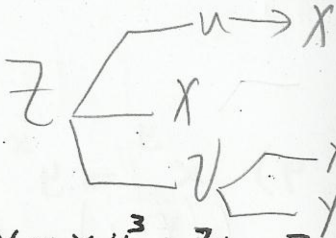
$$f_w = \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial w} = (xz\cos xy)\cdot(v) + (\sin xy)(u)$$

6) $u = x\tan^{-1} xy$, $x = t^2$, $y = re^t$; u_t , u_r



$$u_t = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$
$$= \left(\tan^{-1} xy + \frac{xy}{1+(xy)^2}\right)(2t) + \left(\frac{x^2}{1+x^2y^2}\right)(re^t)$$

$$u_r = \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \left(\frac{x^2}{1+(xy)^2}\right)(e^t)$$



7) $z = u e^{xv} + v e^{-u}$, $u = e^{2x}$, $v = x y^3$; z_x , z_y

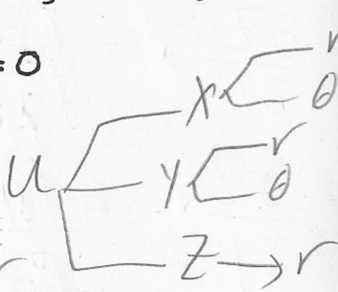
$$Z_x = \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial Z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= v u e^{xv} + e^{xv} \cdot 2e^{2x} + x u e^{xv} \cdot y^3$$

$$Z_y = \frac{\partial Z}{\partial v} \cdot \frac{\partial v}{\partial y} = x u e^{xv} \cdot 3xy^2$$

8) $u = xy + yz + zx$, $x = r\theta$, $y = e^{r\theta}$, $z = r^2$;

$\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$ at $r = 1$, $\theta = 0$



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dr}$$

$$= (y + z)(\theta) + (x + z)(\theta e^{r\theta}) + (y + x)(2r)$$

at $r = 1$, $\theta = 0$ $\Rightarrow$ $\frac{\partial u}{\partial r} = 0 + 0 + 2(x + y) = 2x + 2y$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= (y + z)(r) + (x + z)(r e^{r\theta})$$

At $r = 1$, $\theta = 0$ $\therefore$ $\frac{\partial u}{\partial \theta} = (y + z) + (x + z)$

$$= x + y + z$$

9) $x^{y} = y^{x} + x^{2} - y^{2}$; $\dfrac{dy}{dx}$

$f(x,y) = x^{y} - y^{x} - x^{2} + y^{2} = 0$

$f_x = y\,x^{y-1} - y^{x}\ln y - 2x$

$f_y = x^{y}\ln x - x\,y^{x-1} + 2y$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y\,x^{y-1} - y^{x}\ln y - 2x}{x^{y}\ln x - x\,y^{x-1} + 2y}$$

10) $x^{3} + y^{3} + z^{3} - y^{2} + xz = 8$; $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$

$f(x,y,z) = x^{3} + y^{3} + z^{3} - y^{2} + xz - 8 = 0$

$f_x = 3x^{2} + z$, $f_y = 3y^{2} - 2y$, $f_z = 3z^{2} + x$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{3x^{2} + z}{3z^{2} + x}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{3y^{2} - 2y}{3z^{2} + x}$$

11) If $u = f(x^{2} - y^{2}, 2xy)$, find $\dfrac{\partial^{2} u}{\partial x^{2}}$

let : $x^{2} - y^{2} = r$, $2xy = t$

$\because u = f(r,t) \Longrightarrow$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= f_r \cdot 2x + f_t \cdot 2y$$

$$\frac{\partial^{2} u}{\partial x^{2}} = \frac{\partial u_x}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u_x}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial u_x}{\partial x}$$

$$= (f_{rr}\,2x + f_{tr}\,2y)\,2x + (f_{rt}\cdot 2x + f_{tt}\cdot 2y)\,2y + 2f_r$$

$$z \begin{cases} x \to t \\ y \to t \end{cases}$$

١٢) استخدم قاعدة السلسلة لإيجاد $\frac{d^2z}{dt^2}$ اذا كانت

$$z = x\sin y \;;\; x = e^t \;,\; y = \frac{1}{t}.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$z_t \begin{cases} x \to t \\ y \to t \\ t \end{cases}$$

$$= \sin y \cdot e^t + x\cos y \cdot \left(\frac{-1}{t^2}\right)$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial z_t}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z_t}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial z_t}{\partial t}$$

$$= \left(0 - \frac{\cos y}{t^2}\right)\left(e^t\right) + e^t\cos y + \frac{x\sin y}{t^2} + e^t\sin y + \frac{2x\cos y}{t^3}$$

١٣) اذا كانت $u = f(y+cx) + g(y-cx)$ ، حيث $c \neq 0$ ، فاثبت أن

$$\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial y^2}$$

ارشاد : ضع

$$t = y + cx \;,\; r = y - cx$$

$$u = f(t) + g(r) \;,\; t = y + cx \;,\; r = y - cx$$

$$u \begin{cases} \to t \begin{cases} x \\ y \end{cases} \\ \to r \begin{cases} x \\ y \end{cases} \end{cases}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} = f'(t)\cdot c + g'(r)\cdot(-c)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u_x}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial u_x}{\partial r}\cdot\frac{\partial r}{\partial x} = \left(f''(t)\cdot c\right)\cdot c - \left(g''(r)\cdot c\right)\cdot(-c)$$

$$= c^2\left(f''t + g''r\right) \to (1)$$

باقي حل (٣)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= f'(t)\cdot(1) + g'(r)\cdot(1) = f'(t) + g'(r)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial u_y}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial u_y}{\partial r}\cdot\frac{\partial r}{\partial y} = f''(t) + g''(r) \to (2)$$

من (١)، (٢) ⇐ يتحقق المطلوب

١٤) *إذا كانت

$$u = f(x, y) \;;\quad x = r\cos\theta \;,\; y = r\sin\theta$$

فاثبت أن

$$\text{(i)}\quad u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2}u_\theta^2$$

$$\text{(ii)}\quad u_{xx} + u_{yy} = u_{rr} + \boxed{\cancel{\frac{1}{r}u_r}} + \frac{1}{r^2}u_{\theta\theta}$$

$$u \begin{cases} x \begin{cases} r \\ \theta \end{cases} \\ y \begin{cases} r \\ \theta \end{cases} \end{cases}$$

$$u_r = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x\cdot\cos\theta + f_y\cdot\sin\theta$$

$$u_r \begin{cases} x \begin{cases} r \\ \theta \end{cases} \\ y \begin{cases} r \\ \theta \end{cases} \\ \theta \end{cases}$$

$$u_{rr} = \frac{\partial u_r}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u_r}{\partial y}\cdot\frac{\partial y}{\partial r} = (f_{xx}\cdot\cos\theta)\cos\theta + (f_{yy}\sin\theta)\sin\theta$$

$$u_\theta = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = f_x\cdot -r\sin\theta) + f_y\cdot r\cos\theta)$$

$$u_\theta \begin{cases} x \\ y \\ r \end{cases}$$

$$u_{\theta\theta} = \frac{\partial u_\theta}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u_\theta}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= (f_{xx}\cdot -r\sin\theta)(-r\sin\theta) + (f_{yy}\cdot r\cos\theta)(r\cos\theta)$$

$$= r^2(f_{xx}\sin^2\theta + f_{yy}\cos^2\theta)$$

$$\text{i)}\quad R.H.S = (f_x\cos\theta + f_y\sin\theta)^2 + \frac{1}{r^2}(f_y r\cos\theta - f_x r\sin\theta)^2$$

$$= f_x^2\cos^2\theta + f_y^2\sin^2\theta + 2f_xf_y\cos\theta\sin\theta + f_y^2\cos^2\theta + f_x^2\sin^2\theta - 2f_xf_y\cos\theta\sin\theta$$

$$= f_x^2(\cos^2\theta + \sin^2\theta) + f_y^2(\cos^2\theta + \sin^2\theta) = f_x^2 + f_y^2 = L.H.S$$

$$R.H.S = f_{xx}\cos^2\theta + f_{yy}\sin^2\theta + \frac{1}{\cancel{r^2}}\left[\cancel{r^2}\left(f_{xx}\sin^2\theta + f_{yy}\cos^2\theta\right)\right]$$

$$= f_{xx}\left(\cos^2\theta + \sin^2\theta\right) + f_{yy}\left(\sin^2\theta + \cos^2\theta\right) = f_{xx} + f_{yy}$$

$$= L.H.S$$